2014 – MA

Booklet No. 501870

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Test Paper Code : MA

QUESTION BOOKLET CODE **A**

Reg. No.

Time : 3 Hours

Name :

Maximum Marks : 100

SEAL

GENERAL INSTRUCTIONS

1. This Question-cum-Answer Booklet has **28** pages consisting of Part-I and Part-II.
2. An **ORS** (Optical Response Sheet) is inserted inside the Question-cum-Answer Booklet for filling in the answers of Part-I. Verify that the CODE and NUMBER Printed on the ORS matches with the CODE and NUMBER Printed on the **Question-cum-Answer Booklet.**
3. Based on the performance of Part-I, a certain number of candidates will be shortlisted. Part-II will be evaluated only for those shortlisted candidates.
4. The merit list of the qualified candidates will depend on the performance in both the parts.
5. Write your **Registration Number and Name** on the top right corner of this page as well as on the right hand side of the **ORS**. Also fill the appropriate bubbles for your registration number in the **ORS**.
6. The Question Booklet contains blank spaces for your rough work. No additional sheets will be provided for rough work.
7. **Non-Programmable Calculator is ALLOWED. But clip board, log tables, slide rule, cellular phone and other electronic gadgets are NOT ALLOWED.**
8. The Question-cum-Answer Booklet and the ORS must be returned in its entirety to the Invigilator before leaving the examination hall. **Do not remove any page from this Booklet.**
9. Refer to special instructions/useful data on the reverse of this page.

Instructions for Part-I

10. Part-I consists of **35** objective type questions. The first 10 questions carry **ONE** mark each and the rest 25 questions carry **TWO** marks each.
11. Each question has 4 choices for its answer: (A), (B), (C) and (D). Only **ONE** of the four choices is correct.
12. Fill the correct answer on the left hand side of the included **ORS** by darkening the appropriate bubble with a black ink ball point pen as per the instructions given therein.
13. There will be **negative marks for wrong answers**. For each 1 mark question the negative mark will be 1/3 and for each 2 mark question it will be 2/3.

Instructions for Part-II

14. Part-II has **8** subjective type questions. Answers to this part must be written in blue/black/blue-black ink only. The use of sketch pen, pencil or ink of any other color is not permitted.
15. Do not write more than one answer for the same question. In case you attempt a descriptive question more than once, please cancel the answer(s) you consider wrong. Otherwise, the answer appearing last only will be evaluated.

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Special Instructions / Useful Data

$\mathbb{N}$	: The set of all positive integers
$\mathbb{R}$	: The set of all real numbers
f', f''	: First and second derivatives respectively of a real function f
$\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}$	: Partial derivatives of g with respect to x, y and z respectively
log	: The logarithm to the base e
i, j, k	: Standard unit orthogonal vectors

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IMPORTANT NOTE FOR CANDIDATES

- **Part-I consists of 35 objective type questions. The first ten questions carry *one* mark each and the rest of the objective questions carry *two* marks each. There will be negative marks for wrong answers. For each 1 mark question the negative mark will be 1/3 and for each 2 mark question it will be 2/3.**
- **Write the answers to the objective questions by filling in the appropriate bubble on the left hand side of the included ORS.**
- **Part-II consists of 8 descriptive type questions each carrying *five* marks.**

PART- I

Objective Questions

Q. 1 – Q. 10 carry one mark each.

Q.1 Let $f(x)=\left|x^2-25\right|$ for all $x\in\mathbb{R}$. The total number of points of $\mathbb{R}$ at which f attains a local extremum (minimum or maximum) is

(A) 1 (B) 2 (C) 3 (D) 4

Q.2 The coefficient of $(x-1)^2$ in the Taylor series expansion of $f(x)=xe^x$ $(x\in\mathbb{R})$ about the point $x=1$ is

(A) $\frac{e}{2}$ (B) $2e$ (C) $\frac{3e}{2}$ (D) $3e$

Q.3 Let $f(x,y)=\sum_{k=1}^{10}\left(x^2-y^2\right)^k$ for all $(x,y)\in\mathbb{R}^2$. Then for all $(x,y)\in\mathbb{R}^2$,

(A) $x\frac{\partial f}{\partial x}(x,y)-y\frac{\partial f}{\partial y}(x,y)=0$ (B) $x\frac{\partial f}{\partial x}(x,y)+y\frac{\partial f}{\partial y}(x,y)=0$

(C) $y\frac{\partial f}{\partial x}(x,y)-x\frac{\partial f}{\partial y}(x,y)=0$ (D) $y\frac{\partial f}{\partial x}(x,y)+x\frac{\partial f}{\partial y}(x,y)=0$

Q.4 For $a,\ b,\ c\in\mathbb{R}$, if the differential equation

$$\left(ax^2+bxy+y^2\right)dx+\left(2x^2+cxy+y^2\right)dy=0$$

is exact, then

(A) $b=2,\ c=2a$ (B) $b=4,\ c=2$ (C) $b=2,\ c=4$ (D) $b=2,\ a=2c$

Q.5 If $f(x,y,z)=x^2y+y^2z+z^2x$ for all $(x, y, z)\in\mathbb{R}^3$ and $\nabla=\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}$, then the value of $\nabla\bullet(\nabla\times\nabla f)+\nabla\bullet(\nabla f)$ at $(1, 1, 1)$ is

(A) 0 (B) 3 (C) 6 (D) 9

Q.6 The radius of convergence of the power series $\sum_{n=0}^{\infty}2^{2n}x^{n^2}$ is

(A) $\frac{1}{4}$ (B) 1 (C) 2 (D) 4

Q.7 Let G be a group of order 17. The total number of non-isomorphic subgroups of G is

(A) 1 (B) 2 (C) 3 (D) 17

Q.8 Which one of the following is a subspace of the vector space $\mathbb{R}^3$?

(A) $\{(x,y,z)\in\mathbb{R}^3 : x+2y=0,\ 2x+3z=0\}$

(B) $\{(x,y,z)\in\mathbb{R}^3 : 2x+3y+4z-3=0,\ z=0\}$

(C) $\{(x,y,z)\in\mathbb{R}^3 : x\geq 0,\ y\geq 0\}$

(D) $\{(x,y,z)\in\mathbb{R}^3 : x-1=0,\ y=0\}$

Q.9 Let $T:\mathbb{R}^3\to\mathbb{R}^3$ be the linear transformation defined by $T(x,y,z)=(x+y,\ y+z,\ z+x)$ for all $(x, y, z)\in\mathbb{R}^3$. Then

(A) $\text{rank}(T)=0$, $\text{nullity}(T)=3$ (B) $\text{rank}(T)=2$, $\text{nullity}(T)=1$

(C) $\text{rank}(T)=1$, $\text{nullity}(T)=2$ (D) $\text{rank}(T)=3$, $\text{nullity}(T)=0$

Q.10 Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying $x+\int_0^x f(t)\,dt=e^x-1$ for all $x\in\mathbb{R}$. Then the set $\{x\in\mathbb{R}: 1\leq f(x)\leq 2\}$ is the interval

(A) $[\log 2, \log 3]$ (B) $[2\log 2, 3\log 3]$

(C) $[e-1, e^2-1]$ (D) $[0, e^2]$

Q. 11 – Q. 35 carry two marks each.

Q.11 The system of linear equations

$$\begin{aligned} x - y + 2z &= b_1 \\ x + 2y - z &= b_2 \\ 2y - 2z &= b_3 \end{aligned}$$

is inconsistent when (b_1, b_2, b_3) equals

(A) (2, 2, 0) (B) (0, 3, 2) (C) (2, 2, 1) (D) (2, −1, −2)

Q.12 Let $A = \begin{bmatrix} a & -1 & 4 \\ 0 & b & 7 \\ 0 & 0 & 3 \end{bmatrix}$ be a matrix with real entries. If the sum and the product of all the eigenvalues of A are 10 and 30 respectively, then $a^2 + b^2$ equals

(A) 29 (B) 40 (C) 58 (D) 65

Q.13 Consider the subspace $W = \left\{ (x_1, x_2, \ldots, x_{10}) \in \mathbb{R}^{10} : x_n = x_{n-1} + x_{n-2} \text{ for } 3 \le n \le 10 \right\}$ of the vector space $\mathbb{R}^{10}$. The dimension of W is

(A) 2 (B) 3 (C) 9 (D) 10

Q.14 Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the differential equation

$$x^2 y''(x) - 2xy'(x) - 4y(x) = 0 \text{ for } x \in [1, 10].$$

Consider the Wronskian $W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$. If $W(1) = 1$, then $W(3) - W(2)$ equals

(A) 1 (B) 2 (C) 3 (D) 5

Q.15 The equation of the curve passing through the point $\left(\frac{\pi}{2}, 1\right)$ and having slope $\frac{\sin(x)}{x^2} - \frac{2y}{x}$ at each point (x, y) with $x \neq 0$ is

(A) $-x^2 y + \cos(x) = \frac{-\pi^2}{4}$

(B) $x^2 y + \cos(x) = \frac{\pi^2}{4}$

(C) $x^2 y - \sin(x) = \frac{\pi^2}{4} - 1$

(D) $x^2 y + \sin(x) = \frac{\pi^2}{4} + 1$

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Q.16 The value of $\alpha \in \mathbb{R}$ for which the curves $x^2 + \alpha y^2 = 1$ and $y = x^2$ intersect orthogonally is

(A) -2 (B) $\frac{-1}{2}$ (C) $\frac{1}{2}$ (D) 2

Q.17 Let $x_n = 2^{2n}\left(1-\cos\left(\frac{1}{2^n}\right)\right)$ for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$

(A) does NOT converge
(B) converges to 0
(C) converges to $\frac{1}{2}$
(D) converges to $\frac{1}{4}$

Q.18 Let $\{x_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty}(x_{n+1}-x_n) = c$, where c is a positive real number. Then the sequence $\left\{\frac{x_n}{n}\right\}$

(A) is NOT bounded
(B) is bounded but NOT convergent
(C) converges to c
(D) converges to 0

Q.19 Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series, where $a_n = \frac{(-1)^n n}{2^n}$, $b_n = \frac{(-1)^n}{\log(n+1)}$ for all $n \in \mathbb{N}$. Then

(A) both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent

(B) $\sum_{n=1}^{\infty} a_n$ is absolutely convergent but $\sum_{n=1}^{\infty} b_n$ is conditionally convergent

(C) $\sum_{n=1}^{\infty} a_n$ is conditionally convergent but $\sum_{n=1}^{\infty} b_n$ is absolutely convergent

(D) both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are conditionally convergent

Q.20 The set $\left\{\frac{x^2}{1+x^2} : x \in \mathbb{R}\right\}$ is

(A) connected but NOT compact in $\mathbb{R}$
(B) compact but NOT connected in $\mathbb{R}$
(C) compact and connected in $\mathbb{R}$
(D) neither compact nor connected in $\mathbb{R}$

Q.21 The set of all limit points of the set $\left\{ \frac{2}{x+1} : x \in (-1, 1) \right\}$ in $\mathbb{R}$ is

(A) $[1, \infty)$ (B) $(1, \infty)$ (C) $[-1, 1]$ (D) $[-1, \infty)$

Q.22 Let $S = [0, 1] \cup [2, 3)$ and let $f : S \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 2x & \text{if } x \in [0, 1], \\ 8 - 2x & \text{if } x \in [2, 3). \end{cases}$

If $T = \{ f(x) : x \in S \}$, then the inverse function $f^{-1} : T \to S$

(A) does NOT exist (B) exists and is continuous

(C) exists and is NOT continuous (D) exists and is monotonic

Q.23 Let $f(x) = x^3 + x$ and $g(x) = x^3 - x$ for all $x \in \mathbb{R}$. If f^{-1} denotes the inverse function of f, then the derivative of the composite function $g \circ f^{-1}$ at the point 2 is

(A) $\frac{2}{13}$ (B) $\frac{1}{2}$ (C) $\frac{11}{13}$ (D) $\frac{11}{4}$

Q.24 For all $(x, y) \in \mathbb{R}^2$, let $f(x, y) = \begin{cases} x & \text{if } y = 0, \\ x - y^3 \sin(1/y) & \text{if } y \neq 0. \end{cases}$

Then at the point (0, 0),

(A) f is NOT continuous

(B) f is continuous but NOT differentiable

(C) $\frac{\partial f}{\partial x}$ exists but $\frac{\partial f}{\partial y}$ does NOT exist

(D) f is differentiable

Q.25 For all $(x, y) \in \mathbb{R}^2$, let $f(x, y) = \begin{cases} \frac{x}{|x|}\sqrt{x^2 + y^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

Then $\frac{\partial f}{\partial x}(0, 0) + \frac{\partial f}{\partial y}(0, 0)$ equals

(A) –1 (B) 0 (C) 1 (D) 2

Q.26 Let $f:\mathbb{R}\to\mathbb{R}$ be a function with continuous derivative such that $f(\sqrt{2})=2$ and $f(x)=\lim_{t\to 0}\frac{1}{2t}\int_{x-t}^{x+t} s\,f'(s)\,ds$ for all $x\in\mathbb{R}$. Then $f(3)$ equals

(A) $\sqrt{3}$ (B) $3\sqrt{2}$ (C) $3\sqrt{3}$ (D) 9

Q.27 The value of $\int_{x=0}^{1}\int_{y=0}^{x^2}\int_{z=0}^{y}(y+2z)\,dz\,dy\,dx$ is

(A) $\frac{1}{53}$ (B) $\frac{2}{21}$ (C) $\frac{1}{6}$ (D) $\frac{5}{3}$

Q.28 If C is a smooth curve in $\mathbb{R}^3$ from $(-1, 0, 1)$ to $(1, 1, -1)$, then the value of $\int_C (2xy+z^2)\,dx+(x^2+z)\,dy+(y+2xz)\,dz$ is

(A) 0 (B) 1 (C) 2 (D) 3

Q.29 Let C be the boundary of the region $R=\left\{(x, y)\in\mathbb{R}^2 : -1\le y\le 1, 0\le x\le 1-y^2\right\}$ oriented in the counterclockwise direction. Then the value of $\oint_C y\,dx+2x\,dy$ is

(A) $\frac{-4}{3}$ (B) $\frac{-2}{3}$ (C) $\frac{2}{3}$ (D) $\frac{4}{3}$

Q.30 Let G be a cyclic group of order 24. The total number of group isomorphisms of G onto itself is

(A) 7 (B) 8 (C) 17 (D) 24

Q.31 Let S_n be the group of all permutations on the set $\{1,2,\dots,n\}$ under the composition of mappings. For $n>2$, if H is the smallest subgroup of S_n containing the transposition $(1, 2)$ and the cycle $(1, 2, \dots, n)$, then

(A) $H=S_n$ (B) H is abelian

(C) the index of H in S_n is 2 (D) H is cyclic

Q.32 Let S be the oriented surface $x^2+y^2+z^2=1$ with the unit normal $\mathbf{n}$ pointing outward. For the vector field $\mathbf{F}(x,y,z)=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, the value of $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ is

(A) $\frac{\pi}{3}$ (B) 2π (C) $\frac{4\pi}{3}$ (D) 4π

Q.33 Let $f:(0,\infty)\to\mathbb{R}$ be a differentiable function such that $f'(x^2)=1-x^3$ for all $x>0$ and $f(1)=0$. Then $f(4)$ equals

(A) $\frac{-47}{5}$ (B) $\frac{-47}{10}$ (C) $\frac{-16}{5}$ (D) $\frac{-8}{5}$

Q.34 Which one of the following conditions on a group G implies that G is abelian?

(A) The order of G is p^3 for some prime p

(B) Every proper subgroup of G is cyclic

(C) Every subgroup of G is normal in G

(D) The function $f:G\to G$, defined by $f(x)=x^{-1}$ for all $x\in G$, is a homomorphism

Q.35 Let $S=\left\{x\in\mathbb{R}: x^6-x^5\leq 100\right\}$ and $T=\left\{x^2-2x: x\in(0,\infty)\right\}$. The set $S\cap T$ is

(A) closed and bounded in $\mathbb{R}$

(B) closed but NOT bounded in $\mathbb{R}$

(C) bounded but NOT closed in $\mathbb{R}$

(D) neither closed nor bounded in $\mathbb{R}$

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PART - II

Descriptive Questions

Q. 36 – Q. 43 carry five marks each.

Q.36 Find all the critical points of the function $f:\mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^3 + xy + y^3$ for all $(x, y) \in \mathbb{R}^2$. Also, examine whether the function f attains a local maximum or a local minimum at each of these critical points.

Q.37 Given that there is a common solution to the following equations:

$$\mathbf{P}: y' + 2y = e^x y^2, \quad y(0) = 1,$$

$$\mathbf{Q}: y'' - 2y' + \alpha y = 0,$$

find the value of α and hence find the general solution of $\mathbf{Q}$.

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Q.38 Let $f:\mathbb{R}\to\mathbb{R}$ be a twice differentiable function such that $f\left(\frac{1}{2^n}\right)=0$ for all $n\in\mathbb{N}$. Show that $f'(0)=0=f''(0)$.

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Q.39 Let A be an $n \times n$ matrix with real entries such that $A^2 = A$. If I denotes the $n \times n$ identity matrix, then show that $\text{rank}(A - I) = \text{nullity}(A)$.

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Q.40 Evaluate $\iint_S \frac{xy}{\sqrt{1+2x^2}} dS$, where the surface $S = \left\{ (x,\ y,\ x^2 + y) \in \mathbb{R}^3 :\ 0 \le x \le y,\ x + y \le 1 \right\}$.

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Q.41 Let $f:(0,1)\to\mathbb{R}$ be a differentiable function such that $|f'(x)|\leq 5$ for all $x\in(0,1)$. Show that the sequence $\left\{f\left(\frac{1}{n+1}\right)\right\}$ converges in $\mathbb{R}$.

Q.42 Let H be a subgroup of the group $(\mathbb{R}, +)$ such that $H \cap [-1, 1]$ is a finite set containing a nonzero element. Show that H is cyclic.

Q.43 If K is a nonempty closed subset of $\mathbb{R}$, then show that the set $\{x+y : x \in K,\ y \in [1, 2]\}$ is closed in $\mathbb{R}$.

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Question Number	Marks	
36		
37		
38		
39		
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41		
42		
43		
Total Marks in the Subjective Part		

Total Marks (in words)	:	
Signature of Examiner(s)	:	
Signature of Head Examiner(s)	:	
Signature of Scrutinizer	:	
Signature of Chief Scrutinizer	:	
Signature of Coordinating Head Examiner	:	

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